

GRADIENT INTEGRABILITY AND RIGIDITY RESULTS FOR TWO-PHASE CONDUCTIVITIES IN DIMENSION TWO

VINCENZO NESI, MARIAPIA PALOMBARO, AND MARCELLO PONSIGLIONE

ABSTRACT. This paper deals with higher gradient integrability for σ -harmonic functions u with discontinuous coefficients σ , i.e. weak solutions of $\operatorname{div}(\sigma \nabla u) = 0$. We focus on two-phase conductivities $\sigma : \Omega \subset \mathbb{R}^2 \mapsto \{\sigma_1, \sigma_2\} \subset \mathbb{M}^{2 \times 2}$, and study the higher integrability of the corresponding gradient field $|\nabla u|$. The gradient field and its integrability clearly depend on the geometry, i.e., on the phases arrangement described by the sets $E_i = \sigma^{-1}(\sigma_i)$. We find the optimal integrability exponent of the gradient field corresponding to any pair $\{\sigma_1, \sigma_2\}$ of positive definite matrices, i.e., the worst among all possible microgeometries. We also show that it is attained by so-called exact solutions of the corresponding PDE. Furthermore, among all two-phase conductivities with fixed ellipticity, we characterize those that correspond to the worse integrability.

Keywords: Beltrami system, quasiconformal mappings, elliptic equations, composites, gradient integrability.

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1. INTRODUCTION

Let Ω be a bounded, open and simply connected subset of \mathbb{R}^2 with Lipschitz continuous boundary. We are interested in elliptic equations in divergence form with L^∞ coefficients, specifically,

$$(1.1) \quad \operatorname{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega.$$

Here σ is a matrix valued coefficient, referred to as *conductivity*, and any weak solution $u \in H_{loc}^1(\Omega)$ to the equation is called a σ -harmonic function. The case of discontinuous conductivities σ is particularly relevant in the context of non homogeneous and composite materials. With this motivation, we only assume ellipticity. Denote by $\mathbb{M}^{2 \times 2}$ the space of real 2×2 matrices and by $\mathbb{M}_{sym}^{2 \times 2}$ the subspace of symmetric matrices.

Definition 1.1. *Let $\lambda \in (0, 1]$. We say that $\sigma \in L^\infty(\Omega; \mathbb{M}^{2 \times 2})$ belongs to the class $\mathcal{M}(\lambda, \Omega)$ if it satisfies the following uniform bounds*

$$(1.2) \quad \sigma \xi \cdot \xi \geq \lambda |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega,$$

$$(1.3) \quad \sigma^{-1} \xi \cdot \xi \geq \lambda |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } x \in \Omega,$$

Moreover, we denote by $\mathcal{M}_{sym}(\lambda, \Omega)$ the set $\mathbb{M}_{sym}^{2 \times 2} \cap \mathcal{M}(\lambda, \Omega)$.

The reader may wonder why to use the notion of ellipticity given in Definition 1.1. For an explanation related to its relationship with H -convergence we refer the reader to [2].

It is well known that the gradient of σ -harmonic functions locally belongs to some L^p with $p > 2$. The main goal of this paper is to explore this issue, focusing on two-phase conductivities $\sigma : \Omega \mapsto \{\sigma_1, \sigma_2\} \subset \mathcal{M}$. We will review known results and prove some new one.

Any σ -harmonic function u can be seen as the real part of a complex map $f : \Omega \mapsto \mathbb{C}$ which is a H_{loc}^1 solution to the *Beltrami equation*

$$(1.4) \quad f_{\bar{z}} = \mu f_z + \nu \overline{f_z}, \quad \text{in } \Omega,$$

where the so called complex dilatations μ and ν , both belonging to $L^\infty(\Omega, \mathbb{C})$, are given by

$$(1.5) \quad \mu = \frac{\sigma_{22} - \sigma_{11} - i(\sigma_{12} + \sigma_{21})}{1 + \text{tr } \sigma + \det \sigma}, \quad \nu = \frac{1 - \det \sigma + i(\sigma_{12} - \sigma_{21})}{1 + \text{tr } \sigma + \det \sigma},$$

and satisfy the ellipticity condition

$$(1.6) \quad \|\mu\| + \|\nu\|_{L^\infty} < 1.$$

Let us recall that weak solutions to (1.4) are called quasiregular mappings. They are called quasiconformal if, in addition, they are injective. The ellipticity (1.6) can be expressed by

$$(1.7) \quad \|\mu\| + \|\nu\|_{L^\infty} \leq \frac{K-1}{K+1},$$

for some $K > 1$. The corresponding solutions to (1.4) are called K -quasiregular, and K -quasiconformal if, in addition, they are injective. In 1994, K. Astala [3] proved one of the most important pending conjectures in the field, namely that planar K -quasiregular mappings have Jacobian determinant in $L_{weak}^{K/(K-1)}$. Astala's work represented a benchmark for the issue of determining the optimal integrability exponent which was previously studied in the work of Bojarski [7] and N. Meyers [13].

Summarizing, to any given $\sigma \in \mathcal{M}(\lambda, \Omega)$ one can associate a corresponding pair of complex dilatations via (1.5) and therefore, via the Beltrami equation (1.4) a quasiregular mapping. Therefore, given $\lambda \in [0, 1)$ and given $\sigma \in \mathcal{M}(\lambda, \Omega)$ one can find $K = K(\sigma)$ by using (1.5) and (1.7) in such a way that the σ -harmonic function u , solution to (1.1) is the real part of a K -quasiregular mapping. The Astala regularity results in this context reads as $|\nabla u| \in L_{weak}^{p_K}(\Omega)$, where $p_K := \frac{2K}{K-1}$.

A more refined issue is to determine weighted estimates for the Jacobian determinant of a quasiconformal mapping. A first result in this direction was given in [6]. A much finer

recent result, is given in [4], see formula (1.6). Throughout the present paper we focus on the simpler framework of L^p spaces.

The first question is to determine the best possible (i.e. the minimal) constant $K(\sigma)$ such that if u is σ -harmonic with $\sigma \in \mathcal{M}(\lambda, \Omega)$, then u is the real part of a $K(\sigma)$ -quasiregular mapping. Astala writes in his celebrated paper that his result implies sharp exponents of integrability for the gradient of solutions of planar elliptic pdes of the form (1.1), and he says: “note that the dilation of f and so necessarily the optimal integrability exponent depends in a complicated manner on all the entries of the matrix σ rather than just on its ellipticity”. Alessandrini and Nesi [2], in the process of proving the G -stability of Beltrami equations, made a progress which can be found in their Proposition 1.8. Let us rephrase it here. See also [1] for the estimate (1.9).

Proposition 1.2. *Let $\lambda \in (0, 1]$. Then*

$$(1.8) \quad K_\lambda := \sup_{\sigma \in \mathcal{M}(\lambda, \Omega)} K(\sigma) = \frac{1 + \sqrt{1 - \lambda^2}}{\lambda},$$

$$(1.9) \quad K_\lambda^{sym} := \sup_{\sigma \in \mathcal{M}_{sym}(\lambda, \Omega)} K(\sigma) = \frac{1}{\lambda}.$$

In Section 2.2 we give a simpler and more geometrical proof of Proposition 1.2 based on the real formulation of the Beltrami equation (see Propositions 2.2 and 2.3). In [2], pp. 63, the authors noticed that, the supremum in (1.8) is attained on specific non symmetric matrices. As a straightforward corollary, in [2] the authors write the version of Astala’s theorem which is adequate for matrices belonging to $\mathcal{M}(\lambda, \Omega)$ that we recall here in an informal way. Any σ -harmonic function with $\sigma \in \mathcal{M}(\lambda, \Omega)$ satisfies the property $|\nabla u| \in L_{weak}^{p_{K_\lambda}}$, where K_λ is given by (1.8) and $p_{K_\lambda} := \frac{2K_\lambda}{K_\lambda - 1}$. This has to be compared with the version that holds true assuming a priori that $\sigma \in \mathcal{M}_{sym}(\lambda, \Omega)$. In that case K_λ can be replaced by K_λ^{sym} defined in (1.9). Optimality in the latter case was proved by Leonetti and Nesi [12] which began their work using the bound (1.9) which had been already observed in Alessandrini and Magnanini [1]. Optimality means that there exists $\sigma \in \mathcal{M}_{sym}(\lambda, \Omega)$ for which the estimate $|\nabla u| \in L_{weak}^{p_{K_\lambda}}$ is sharp.

Later there has been a number of increasingly refined results showing optimality of Astala’s theorem for a different class of symmetric matrices σ . Specifically Faraco [8] treats the case of two isotropic materials, i.e. when σ takes values only in the set of two matrices of the form $\{KI, \frac{1}{K}I\}$, with I the identity matrix, which was originally conjectured to be optimal for the exponent $\frac{2K}{K-1}$ by Milton [15]. In a further advance a more refined version was given in [5], where the authors proved optimality in the stronger sense of exact solutions.

However the original question implicitly raised by Astala was apparently forgotten. In this paper we go back to that and we prove optimality for a generic two-phase matrix field $\sigma \in \mathcal{M}(\lambda, \Omega)$. To describe our approach let us first recall that when σ is smooth, the corresponding σ -harmonic function is necessarily smooth and hence with bounded gradient. So the issue of higher exponent of integrability is really related to discontinuous coefficients. The simplest class of examples is when one has a conductivity taking only two values. We therefore ask the following questions. Given two positive definite matrices, σ_1 and σ_2 , consider the class of matrices $\sigma \in \mathcal{M}(\lambda, \Omega)$ of the special form $\sigma(x) = \sigma_1 \chi_{E_1} + \sigma_2 \chi_{E_2}$, where $\{E_1, E_2\}$ is a measurable partition of Ω and χ_{E_i} denotes the characteristic function of the set E_i . In the jargon of composite materials this is called a two-phase composite. What is the best

possible information one can extrapolate from Astala's Theorem? As already explained, to the ellipticity λ of σ there corresponds a suitable constant $K(\sigma)$ in the Beltrami equation. We are naturally led to the following related question: given $\mu, \nu \in L^\infty(\Omega; \mathbb{C})$ satisfying (1.7) with $K(\mu, \nu) > 1$, is it possible to transform μ and ν , by a suitable change of variables, specifically, by affine transformations, in order to decrease K and thus gain a better integrability for the solution of the transformed Beltrami equation? The key observation here is that the summability of solutions of the Beltrami equation is invariant under such transformations, while $K(\mu, \nu)$ is not. It is then well defined the minimal Beltrami constant K^{min} attainable under such transformations. In Proposition 5.4 we find an explicit formula for such K^{min} in terms of all the entries of σ_1 and σ_2 . Moreover, K^{min} gives a sharp measure of the integrability properties of solutions to (1.1). This is stated in Theorem 5.1, which, for the reader's convenience, we reformulate here in a more informal way.

Theorem 1.3. *i) Let $\sigma \in \mathcal{M}(\lambda, \Omega)$ with $\sigma \in \{\sigma_1, \sigma_2\}$. Every σ -harmonic function u satisfies $\nabla u \in L_{loc}^p(\Omega)$ for every $p \in [2, p_{K^{min}})$. ii) There exist $\sigma \in \mathcal{M}(\lambda, \Omega)$ with $\sigma \in \{\sigma_1, \sigma_2\}$ and a σ -harmonic function with affine boundary conditions such that, for every ball $B \subset \Omega$*

$$(1.10) \quad \int_B |\nabla u|^{p_{K^{min}}} dx = \infty.$$

A key step to prove Theorem 1.3 is to prove the optimality of Astala's Theorem for a new class of symmetric conductivities, specifically, for matrices of the form

$$(1.11) \quad \sigma = \chi_{E_1} \text{diag}(S_1, \lambda^{-1}) + \chi_{E_2} \text{diag}(S_2, \lambda), \quad \text{with } \lambda \leq S_1, S_2 \leq \lambda^{-1},$$

thus generalizing the isotropic case $S_1 = \lambda^{-1}$, $S_2 = \lambda$, considered in [5] and [3].

As a corollary of Theorem 1.3, we prove that the bound (1.8) for non symmetric matrices too is optimal. Indeed, there exists $\sigma \in \mathcal{M}(\lambda, \Omega)$ of the form

$$(1.12) \quad \sigma = \chi_{E_1} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \chi_{E_2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with } a = \lambda, b = \pm \sqrt{1 - \lambda^2}$$

and a σ -harmonic function u such that the bound $\nabla u \in L_{weak}^{p_{K_\lambda}}$ is sharp (see Theorem 5.2).

Finally, a natural question, both in the symmetric and in the non symmetric case, is whether there are other two-phase critical coefficients, that is to say, two-phase coefficients σ for which the bounds in Proposition 1.2 are attained and optimal in the sense of (1.10). In Theorem 5.3 we give a complete answer to this question, characterizing all the critical conductivities with fixed ellipticity. In the symmetric case, the critical conductivities are given (up to rotations) exactly by those in (1.11) (for suitable partitions E_1, E_2). In the non symmetric case, the only critical conductivities are as in (1.12).

We remark that one can find optimal microgeometries for σ 's which are not two-phase. The simplest example is given by a "polycrystal" like in the first example given in Leonetti and Nesi [12]. In that case σ is symmetric, the eigenvalues are λ and λ^{-1} but the eigenvectors change from point to point.

2. MORE ABOUT σ -HARMONIC FUNCTIONS AND THE BELTRAMI SYSTEM

In the present section we review some well-known connections between σ -harmonic functions and the Beltrami system which we use in the rest of the paper. We refer the interested reader to [2] for a more detailed presentation of the argument.

2.1. Complex vs real formulation of a Beltrami system. Consider the Beltrami equation (1.4). It can be rewritten in the equivalent form

$$(2.1) \quad Df^t H Df = G \det Df,$$

where G and H are real matrix fields depending on μ and ν . Specifically,

$$(2.2) \quad \begin{aligned} G &= \frac{1}{d} \begin{pmatrix} |1 + \mu|^2 - |\nu|^2 & 2\Im(\mu) \\ 2\Im(\mu) & |1 - \mu|^2 - |\nu|^2 \end{pmatrix}, \\ H &= \frac{1}{d} \begin{pmatrix} |1 - \nu|^2 - |\mu|^2 & -2\Im(\nu) \\ -2\Im(\nu) & |1 + \nu|^2 - |\mu|^2 \end{pmatrix}, \end{aligned}$$

where

$$d = \sqrt{(1 - (|\nu| - |\mu|)^2)(1 - (|\nu| + |\mu|)^2)}.$$

We will refer to (1.4) as well as to (2.1) as the Beltrami system. Let $SL(2)$ be the subset of $\mathbb{M}^{2 \times 2}$ of the invertible matrices with determinant one, and let $SL_{sym}(2) = \mathbb{M}_{sym}^{2 \times 2} \cap SL(2)$. Notice that G and H belong to $SL_{sym}(2)$ and they are positive definite. In fact injective solutions to (2.1) have a very neat geometrical interpretation. They are mapping $f : \Omega \rightarrow \Omega'$ which are conformal, i.e., they preserves angles, provided one uses the right scalar products, namely the one induced by G in Ω and H in Ω' . This interpretation has many consequences. We will get back to this point later in the paper. Inversion of the above formulas yields

$$\mu = \frac{G_{11} - G_{22} + 2iG_{12}}{G_{11} + G_{22} + H_{11} + H_{22}}, \quad \nu = \frac{H_{22} - H_{11} - 2iH_{12}}{G_{11} + G_{22} + H_{11} + H_{22}}.$$

By combining (2.2) and (1.5) we obtain a formula for G and H as functions of σ ,

$$(2.3) \quad G(\sigma) = \frac{1}{\sqrt{\det \sigma^S}} \begin{pmatrix} \sigma_{22} & -\frac{\sigma_{12} + \sigma_{21}}{2} \\ -\frac{\sigma_{12} + \sigma_{21}}{2} & \sigma_{11} \end{pmatrix}, \quad H(\sigma) = \frac{1}{\sqrt{\det \sigma^S}} \begin{pmatrix} \det \sigma & \frac{-\sigma_{12} + \sigma_{21}}{2} \\ \frac{-\sigma_{12} + \sigma_{21}}{2} & 1 \end{pmatrix},$$

where $\sigma^S = \frac{\sigma + \sigma^T}{2}$. Inversion of (2.3) gives

$$(2.4) \quad \sigma = \frac{1}{H_{22}} (G^{-1} + H_{12}J)$$

where

$$(2.5) \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, we can express σ as a function of μ, ν inverting the algebraic system (1.5),

$$(2.6) \quad \sigma = \begin{pmatrix} \frac{|1 - \mu|^2 - |\nu|^2}{|1 + \nu|^2 - |\mu|^2} & \frac{2\Im(\nu - \mu)}{|1 + \nu|^2 - |\mu|^2} \\ \frac{-2\Im(\nu + \mu)}{|1 + \nu|^2 - |\mu|^2} & \frac{|1 + \mu|^2 - |\nu|^2}{|1 + \nu|^2 - |\mu|^2} \end{pmatrix}.$$

Let us clarify the relationship between the Beltrami equation and σ -harmonic maps. Given positive definite matrices G and H in $L^\infty(\Omega; SL_{sym}(2))$, let $f = (u, v)$ be solution to (2.1). Then, the function u is σ -harmonic, with σ defined by (2.4). Conversely, given σ satisfying the ellipticity conditions (1.2)-(1.3) and given a σ -harmonic function u , the map $f := (u, v)$ solves (2.1), where G and H are defined by (2.3), v is such that

$$(2.7) \quad J^T \nabla v = \sigma \nabla u,$$

and J^T is the transpose of J defined in (2.5). The function v is called *stream function* of u , and is defined up to additive constants. Moreover, $\|\nabla f\|_{L^p}$ is finite if and only if $\|\nabla u\|_{L^p}$ is finite.

2.2. Different formulations of ellipticity and higher gradient integrability. Here we introduce classical notions of ellipticity for elliptic and Beltrami equations, and we recall the fundamental summability results due to Astala [3] and some of its consequences due to Leonetti and Nesi [12]. From now on, we will always assume that the values of μ, ν, G, H and σ are related according to (1.5) and (2.2).

The ellipticity corresponding to any pair $\mu, \nu \in L^\infty(\Omega; \mathbb{C})$ satisfying (1.6) is the positive constant $k(\mu, \nu)$ defined by

$$(2.8) \quad k(\mu, \nu) := \frac{\|\mu\| + \|\nu\|}{\|\mu\| - \|\nu\|}.$$

An alternative measure of ellipticity, that will be most convenient in our analysis, is provided by the following quantity

$$(2.9) \quad K(\mu, \nu) := \frac{1 + k(\mu, \nu)}{1 - k(\mu, \nu)}.$$

Having in mind (2.2), we define $k(G, H)$ and $K(G, H)$ in the obvious way, i.e.,

$$(2.10) \quad k(G, H) = k(\mu, \nu), \quad K(G, H) = K(\mu, \nu),$$

and whenever no confusion may arise, we will omit the dependance on their argument. In the next proposition we give a more explicit formula for such ellipticity. We will denote by $g(x)$ e $h(x)$ the maximum eigenvalue of $G(x)$ and $H(x)$, respectively.

Proposition 2.1. *Let $G, H \in L^\infty(\Omega; SL_{sym}(2))$ be positive definite. Then*

$$(2.11) \quad K = \|gh\|_{L^\infty(\Omega)}.$$

Proof. A direct computation shows that the maximum eigenvalues of G and H are given by

$$g = \frac{\sqrt{(1 - |\nu| + |\mu|)(1 + |\nu| + |\mu|)}}{\sqrt{(1 + |\nu| - |\mu|)(1 - |\nu| - |\mu|)}}, \quad h = \frac{\sqrt{(1 + |\nu| - |\mu|)(1 + |\nu| + |\mu|)}}{\sqrt{(1 - |\nu| + |\mu|)(1 - |\nu| - |\mu|)}}.$$

Therefore $gh = \frac{1 + |\mu| + |\nu|}{1 - (|\mu| + |\nu|)}$, which yields

$$\|gh\|_{L^\infty} = \frac{1 + \|\mu\| + \|\nu\|}{1 - \|\mu\| - \|\nu\|} = \frac{1 + k}{1 - k} = K.$$

□

Next, we relate the ellipticity bounds for the second order elliptic operator (1.1) with the ellipticity of the associated Beltrami equation. Following the notation of (2.10), we set $K(\sigma) := K(G, H)$, where G, H and σ are related by (2.3)-(2.4). The following result has been proved in [12] and [2]; for the reader's convenience, we give here a proof based on Proposition 2.1.

Proposition 2.2. *Let $\lambda \in (0, 1]$. For each $\sigma \in \mathcal{M}(\lambda, \Omega)$ we have*

$$(2.12) \quad K(\sigma) \leq \frac{1 + \sqrt{1 - \lambda^2}}{\lambda}.$$

If in addition σ is symmetric, then

$$(2.13) \quad K(\sigma) \leq \frac{1}{\lambda}.$$

Proof. Let λ_1, λ_2 be the eigenvalues of σ^S , with $\lambda_1 \leq \lambda_2$. Then, from the assumption $\sigma \in \mathcal{M}(\lambda, \Omega)$ and the relationship

$$(\sigma^{-1})^S = \frac{\det \sigma^S}{\det \sigma} (\sigma^S)^{-1},$$

it follows

$$(2.14) \quad \lambda_2 \geq \lambda_1 \geq \lambda,$$

$$(2.15) \quad \frac{\det \sigma^S}{\lambda_2 \det \sigma} = \frac{\lambda_1}{\det \sigma} \geq \lambda.$$

Next let g and h be the largest eigenvalue of G and H respectively. By (2.4), it is readily seen that

$$\sigma^S = \frac{1}{H_{22}} G^{-1}.$$

and hence

$$(2.16) \quad g = \frac{1}{H_{22}} \frac{1}{\lambda_1} = \frac{\sqrt{\det \sigma^S}}{\lambda_1}.$$

From (2.3) it follows

$$(2.17) \quad h + \frac{1}{h} = \frac{1}{\sqrt{\det \sigma^S}} (\det \sigma + 1).$$

Set $P := \frac{\det \sigma + 1}{\sqrt{\det \sigma^S}}$. Solving (2.17) and choosing the root which is bigger than one, yields

$$(2.18) \quad h = \frac{P + \sqrt{P^2 - 4}}{2}.$$

Then, using (2.16)-(2.18) and the inequalities (2.14)-(2.15), we obtain the following upper bound for gh

$$\begin{aligned} gh &= \frac{1}{2\lambda_1} \left[\det \sigma + 1 + \sqrt{(\det \sigma + 1)^2 - 4 \det \sigma^S} \right] \\ &\leq \frac{1}{2\lambda_1} \left[\frac{\lambda_1}{\lambda} + 1 + \sqrt{\left(\frac{\lambda_1}{\lambda} + 1 \right)^2 - 4\lambda_1^2} \right] \\ &= \frac{1}{2} \left(\frac{1}{\lambda} + \frac{1}{\lambda_1} + \sqrt{\frac{1}{\lambda^2} + \frac{1}{\lambda_1^2} + \frac{2}{\lambda\lambda_1} - 4} \right) \\ &\leq \frac{1}{2} \left(\frac{2}{\lambda} + \sqrt{\frac{4}{\lambda^2} - 4} \right) \\ &= \frac{1 + \sqrt{1 - \lambda^2}}{\lambda}. \end{aligned}$$

Now suppose that σ is symmetric and denote by λ_1 and λ_2 its eigenvalues, with $\lambda_1 \leq \lambda_2$. Since $\sigma \in \mathcal{M}(\lambda, \frac{1}{\lambda}, \Omega)$, we have

$$(2.19) \quad \lambda \leq \lambda_1 \leq \lambda_2 \leq \frac{1}{\lambda}.$$

Formula (2.3) reduces itself to

$$G = \sqrt{\det \sigma} \sigma^{-1}, \quad H = \frac{1}{\sqrt{\det \sigma}} \begin{pmatrix} \det \sigma & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$(2.20) \quad g = \frac{1}{\lambda_1} \sqrt{\det \sigma}, \quad h = \frac{1}{\sqrt{\det \sigma}} \max\{\lambda_1 \lambda_2, 1\}.$$

In the case when $\lambda_1 \lambda_2 \leq 1$, we find

$$K = \left\| \frac{1}{\lambda_1} \right\|_{L^\infty} \leq \frac{1}{\lambda}.$$

If otherwise $\lambda_1 \lambda_2 \geq 1$, we have

$$K = \|\lambda_2\|_{L^\infty} \leq \frac{1}{\lambda}.$$

□

In the next Proposition we look at conductivities σ attaining the bounds (2.12) and (2.13).

Proposition 2.3. *Let $\sigma \in \mathcal{M}(\lambda, \Omega)$ for some $\lambda \in (0, 1]$. Then the bound (2.12) is attained if and only if on a set of positive measure there holds*

$$(2.21) \quad \sigma = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \text{with } a = \lambda, b = \pm \sqrt{1 - \lambda^2}.$$

Moreover, if σ is symmetric (2.13) is attained if and only if either (1.2) or (1.3) is attained on a set of positive measure.

Proof. Keeping the notation introduced in the proof of Proposition 2.2, one can see that the bound (2.12) is attained if and only if the inequalities (2.14)-(2.15) hold as equalities, namely,

$$\lambda_2 = \lambda_1 = \lambda, \quad \frac{\lambda_1}{\det \sigma} = \lambda.$$

It is readily seen that this is equivalent to (2.21). The symmetric case is left to the reader. □

We now recall the higher integrability results for gradients of solutions to (1.1) and (1.4). For $K > 1$, set $p_K := \frac{2K}{K-1}$. We start with the celebrated result in [3].

Theorem 2.4. *Let $f \in H_{loc}^1(\Omega; \mathbb{C})$ be solution to (1.4) with $K(\mu, \nu) > 1$. Then*

$$\nabla f \in L_{loc}^p(\Omega) \quad \forall p \in [2, p_{K(\mu, \nu)}).$$

Recall that K_λ and K_λ^{sym} are defined by (1.8) and (1.9), respectively. A straightforward computation yields

$$(2.22) \quad p_{K_\lambda} = \frac{2 + 2\sqrt{1 - \lambda^2}}{1 - \lambda + \sqrt{1 - \lambda^2}}, \quad p_{K_\lambda^{sym}} = \frac{2}{1 + \lambda}.$$

As a consequence of Proposition 2.2 and Theorem 2.4, we obtain the following result which was proved in [12], [2].

Theorem 2.5. *Let $\sigma \in \mathcal{M}(\lambda, \Omega)$ for some $\lambda \in (0, 1]$. Then, any solution $u \in H_{\text{loc}}^1(\Omega)$ to (1.1) satisfies*

$$\nabla u \in L_{\text{loc}}^p(\Omega) \quad \forall p \in [2, p_{K_\lambda}),$$

and, if $\sigma \in L^\infty(\Omega; \mathbb{M}_{\text{sym}}^{2 \times 2})$,

$$\nabla u \in L_{\text{loc}}^p(\Omega) \quad \forall p \in [2, p_{K_\lambda^{\text{sym}}}),$$

where p_{K_λ} and $p_{K_\lambda^{\text{sym}}}$ are given in (2.22).

We are now ready to perform linear change of variables both in the domain and in the target space. It will be convenient to work with the real formulation of the equation (2.1). Let $A, B \in SL(2)$ and set

$$(2.23) \quad \tilde{f}(x) := A^{-1}f(Bx), \quad \tilde{G}(x) := B^t G(Bx)B, \quad \tilde{H}(x) := A^t H(Bx)A.$$

A straightforward computation shows that, whenever $f : \Omega \mapsto \mathbb{R}^2$ is solution to (2.1), \tilde{f} solves

$$(2.24) \quad D\tilde{f}^t \tilde{H} D\tilde{f} = \tilde{G} \det D\tilde{f}.$$

Clearly \tilde{f} enjoys the same integrability properties as f . This motivates the following definition,

$$(2.25) \quad K^{\min}(G, H) := \min_{A, B \in SL(2)} \|g(A, B)h(A, B)\|_{L^\infty},$$

where $g(A, B)$ and $h(A, B)$ denote the maximum eigenvalue of \tilde{G} and \tilde{H} , respectively. Remark that $g(A) \geq c\|A\|^2$, $h(B) \geq c\|B\|^2$. Therefore, the minimum in (2.25) is attained. Recalling (2.11), a straightforward generalization of Theorem 2.4 leads to the following result.

Proposition 2.6. *Let $G, H \in SL_{\text{sym}}(2)$ and let $K^{\min}(G, H)$ be defined as in (2.25). Then any $f \in H_{\text{loc}}^1(\Omega; \mathbb{R}^2)$ solution to (2.1) satisfies*

$$\nabla f \in L_{\text{loc}}^p(\Omega) \quad \forall p \in [2, p_{K^{\min}(G, H)}).$$

Remark 2.7. *From the point of view of σ -harmonic maps, Proposition 2.6 may be rephrased by saying that any solution $u \in H_{\text{loc}}^1(\Omega)$ to (1.1), satisfies*

$$\nabla u \in L_{\text{loc}}^p(\Omega) \quad \forall p \in [2, p_{K^{\min}(\sigma)}),$$

where $K^{\min}(\sigma)$ is defined in the obvious way, i.e., $K^{\min}(\sigma) := K^{\min}(G, H)$, and G, H and σ are related by (2.3).

3. EXAMPLES OF WEAK SOLUTIONS WITH CRITICAL INTEGRABILITY PROPERTIES

In [8], [5], the authors exhibit an example of weak solution to (1.1) with critical integrability properties. In their construction the essential range of σ consists of only two isotropic matrices, namely, $\sigma : \Omega \mapsto \{K^{-1}I, KI\}$ with $K > 1$. In this section we generalize their construction to the case

$$(3.1) \quad \sigma_1 := \text{diag}(K, S_1), \quad \sigma_2 := \text{diag}(K^{-1}, S_2), \quad \frac{1}{K} \leq S_i \leq K,$$

thus proving optimality of Astala's theorem for the whole class of matrices above. In Section 5 we will show that such class cannot be further enlarged.

We will need the following definition.

Definition 3.1. The family of *laminates of finite order* is the smallest family of probability measures $\mathcal{L}(\mathbb{M}^{2 \times 2})$ on $\mathbb{M}^{2 \times 2}$ such that

- (i) $\mathcal{L}(\mathbb{M}^{2 \times 2})$ contains all Dirac masses;
- (ii) if $\sum_{i=1}^n \alpha_i \delta_{A_i} \in \mathcal{L}(\mathbb{M}^{2 \times 2})$ and $A_1 = \alpha B + (1 - \alpha)C$ with $\text{rank}(B - C) = 1$, then the probability measure $\sum_{i=2}^n \alpha_i \delta_{A_i} + \alpha_1(\alpha B + (1 - \alpha)C)$ is also contained in $\mathcal{L}(\mathbb{M}^{2 \times 2})$.

Given $\nu \in \mathcal{L}(\mathbb{M}^{2 \times 2})$, we define the barycenter $\bar{\nu}$ of ν as

$$\bar{\nu} := \int_{\mathbb{M}^{2 \times 2}} M d\nu(M).$$

Theorem 3.2. *Let σ_1, σ_2 be as in (3.1). There exists a measurable matrix field $\sigma : \Omega \rightarrow \{\sigma_1, \sigma_2\}$ such that the solution $u \in H^1(\Omega)$ to*

$$(3.2) \quad \begin{cases} \text{div}(\sigma \nabla u) = 0 & \text{in } \Omega \\ u(x) = x_1 & \text{on } \partial\Omega \end{cases}$$

satisfies for every ball $B \subset \Omega$

$$\int_B |\nabla u|^{p_K} dx = \infty.$$

The proof follows the strategy in [5, Theorem 3.13], where the result is proved for $\sigma_1 = KI$, $\sigma_2 = K^{-1}I$. Here the main difference is that we work with coefficients that are not isotropic. For the reader's convenience we shortly reproduce the arguments of [5] pointing out the essential modifications.

Step 1 (*Reformulation of (3.2) as a differential inclusion*). Recall that u is solution to (3.2) if and only if $u = f_1$ where $f = (f_1, f_2)$ is solution to the associated Beltrami equation. It is easily checked that, for σ of the form (3.1), the latter condition is equivalent to

$$(3.3) \quad Df \in E := E_1 \cup E_2,$$

where

$$E_1 = \left\{ \begin{pmatrix} T \\ J\sigma_1 T \end{pmatrix}, T \in \mathbb{R}^2 \right\}, \quad E_2 = \left\{ \begin{pmatrix} T \\ J\sigma_2 T \end{pmatrix}, T \in \mathbb{R}^2 \right\}.$$

The goal is to find a solution $f \in H^1(\Omega; \mathbb{R}^2)$ to the differential inclusion (3.3) satisfying in addition the boundary condition $f_1(x) = x_1$ on $\partial\Omega$.

Next we define a setting where to apply the Baire category method. Fix $\delta > 0$ such that

$$(3.4) \quad \delta < \left(\frac{(1 - 1/K)(K - 1)}{4 \max\{S_1, S_2\} K^2} \right)^{\frac{1}{2}},$$

and let

$$(3.5) \quad \tilde{E} := E \cap \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathbb{M}^{2 \times 2} : |a_{12}| < \delta a_{11} \right\}.$$

Notice that the introduction of the small parameter δ enforces the solutions to have gradient pointing in a direction relatively close to I . This property hides the anisotropy of the coefficients σ_i , and allows us to follow the strategy of [5]. Define \mathcal{U} as the interior of the quasiconvex hull of \tilde{E} (defined as the set of range of weak limits in L^2 of solutions to (3.3)). The following characterization of $\overline{\mathcal{U}}$ holds

$$\tilde{E}^{lc,1} = \overline{\mathcal{U}} = \tilde{E}^{pc},$$

where $\tilde{E}^{lc,1}$ and \tilde{E}^{pc} denote the first lamination hull and the polyconvex hull of \tilde{E} , respectively. We refer to [5, Lemma 3.5] for the proof of the identity above and for the notion of first lamination hull and polyconvex hull.

Set

$$X_0 = \{f \in W^{1,\infty}(\bar{\Omega}; \mathbb{R}^2) : f \text{ piecewise affine, } Df \in \mathcal{U} \text{ a.e., } f|_{\partial\Omega} = x\},$$

let X be its closure in the weak topology of H^1 , and denote by (X, w) the set X endowed with the weak topology w of H^1 . Remark that $I \in \mathcal{U}$ and therefore the set X is not empty as it contains the map $f(x) = x$.

Step 2 (*Existence of solutions by the Baire category method*). The existence of solutions to the differential inclusion is proved by an application of the Baire category method, and is based on the fact that the gradient operator $D : X \mapsto L^2(\Omega; \mathbb{M}^{2 \times 2})$ is a Baire-1 mapping, i.e., the pointwise limit of continuous mappings. We refer to [10, page 57] and references therein for further clarifications on this subject. The existence result is stated in the next theorem. We refer to [5, Lemma 3.7] for its proof.

Theorem 3.3. *The space (X, w) is compact and metrizable. Each $f \in X$ satisfies $f \in \bar{\mathcal{U}}$ and $f|_{\partial\Omega} = x$. The metric d on X is equivalent to the metrics induced by the L^2 and L^∞ norms. Moreover, the points of continuity of the map $D : (X, w) \rightarrow L^2(\Omega; \mathbb{M}^{2 \times 2})$ form a residual set in (X, w) . Finally, any point of continuity $f \in X$ of D satisfies $Df \in E_1 \cup E_2$.*

We deduce that the set of solutions to the differential inclusion (3.3) is residual in (X, w) . The proof of Theorem 3.2 is then a consequence of Theorem 3.3 and of the following theorem.

Theorem 3.4. *The set*

$$\left\{ f \in X : \int_B |Df|^{p_K} = +\infty \text{ for all balls } B \subset \Omega \right\}$$

is residual in X .

Theorem 3.4 is proved following the same strategy of the proof of Corollary 3.12 in [5]. We recall that in [5] the isotropic case $S_1 = K$, $S_2 = 1/K$ is considered. In the present setting the proof is identical except for the proof of a key ingredient (namely, [5, Proposition 3.10]). Therefore, we only state and proof such result in Lemma 3.5 below. For this purpose it is convenient to introduce some notation. Given a matrix $A = (a_{ij}) \in \mathbb{M}^{2 \times 2}$, we denote by A_d and A_a its diagonal and anti-diagonal part, namely

$$(3.6) \quad A_d := \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \quad A_a := \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}.$$

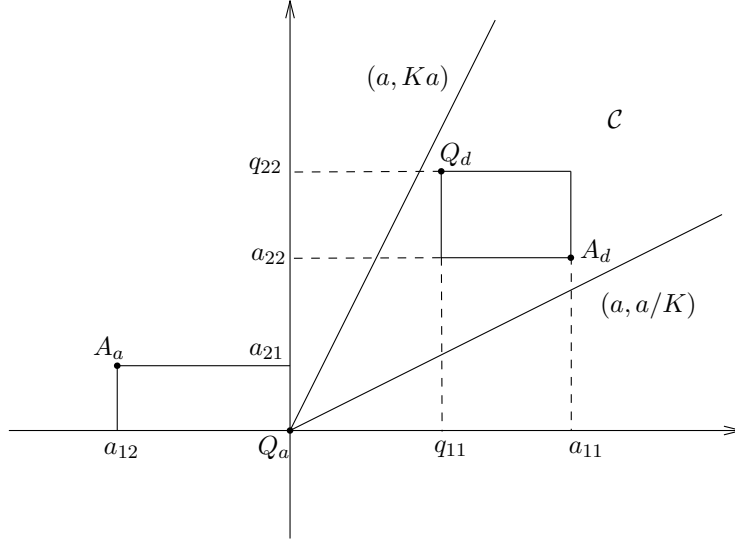
Moreover we will identify A_d and A_a with points of \mathbb{R}^2 : $A_d = (a_{11}, a_{22})$, $A_a = (a_{12}, a_{21})$. Finally, we denote by \mathcal{C} the following cone of \mathbb{R}^2 .

$$(3.7) \quad \mathcal{C} = \{t(a, a/K) + (1-t)(a, Ka) : t \in (0, 1), a \in \mathbb{R}^+\}.$$

Lemma 3.5. *Every $A \in \mathcal{U}$ is the barycenter of a sequence of laminates of finite order $\nu_n \in \mathcal{L}$ such that $\text{supp } \nu_n \subset \mathcal{U}$ and*

$$(3.8) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{M}^{2 \times 2}} |M|^{p_K} d\nu_n(M) = \infty.$$

Proof. The proof of Lemma 3.5 follows the strategy of the proof of [5, Proposition 3.10], where the particular case of $S_1 = K$ and $S_2 = 1/K$ is considered. In [5] it is first showed that the identity matrix is the barycenter of a sequence of laminates of finite order satisfying

FIGURE 1. The rank-one connected points A and Q .

(3.8) and with support on $\partial\mathcal{C}$, where \mathcal{C} is the cone defined by (3.7). The proof is based on the construction of the so-called staircase laminates, which was originally made in [8]. Then, they extend the result to all other matrices by using the conformal invariance of the quasiconvex hull. In our case \mathcal{U} does not enjoy conformal invariance, due to the anisotropy of the coefficients σ_i . Therefore, we have to proceed in a different way.

By slightly modifying the staircase construction in [8], [5] (in fact only a finite number of steps at the beginning of the staircase) one can easily show that each point in \mathcal{C} can be obtained as the barycenter of a sequence of laminates of finite order, satisfying (3.8) and with support on $\partial\mathcal{C}$. Moreover, by a suitable shift of the support, one can obtain that these measures have support in the interior of the cone \mathcal{C} .

Now let $A = (a_{ij}) \in \mathcal{U}$. We claim that A is rank-one connected to a diagonal matrix $Q = (q_{ij}) = Q_d \in \mathcal{C}$ and we conclude the proof. Arguing as in [5, Remark 3.6] it is easy to show that $Q \in \mathcal{U}$ (that is to say, Q belongs to the interior of the quasiconvex hull), and that A belongs to a suitable segment $[P, Q]$ still contained in \mathcal{U} , i.e., $A = \tau P + (1 - \tau)Q$ for some $\tau \in (0, 1)$. Since $Q \in \mathcal{C}$, Q is the barycenter of a sequence of laminates $\nu_n = \sum \lambda_j \delta_{A_j}$ supported in \mathcal{U} and satisfying (3.8). The required laminates can then be defined as

$$\tilde{\nu}_n = \tau \delta_P + (1 - \tau) \sum \lambda_j \delta_{A_j}.$$

We conclude by proving the claim. From (3.5) it follows that $A_d \in \mathcal{C}$. The condition of rank-one connectedness reads as

$$(3.9) \quad (a_{11} - q_{11})(a_{22} - q_{22}) = a_{21}a_{12}.$$

This is equivalent to the fact that the two rectangles with sides parallel to the axis and diagonal $Q_d A_d$ and $Q_a A_a$ have the same signed area (see Figure 1). Notice that the sign of the areas is given by the sign of the slope of $Q_d A_d$ and of $Q_a A_a$, respectively. Define $h(A_d, Q_d)$ as the signed area of the corresponding rectangle and remark that it is a continuous function.

Given A , the problem is to find Q_d such that

$$(3.10) \quad h(A_d, Q_d) = a_{21}a_{12}.$$

Notice that

$$\{h(A_d, Q_d), Q_d \in \overline{\mathcal{C}}\} = [m, \infty)$$

for a suitable negative $m < 0$ depending on A_d . Therefore, If $a_{21}a_{12} > 0$ we can always solve (3.10). Assume instead that $a_{21}a_{12} < 0$ like in Figure 1. Let $\tilde{h}(A_d)$ be the infimum of h over Q_d . For a fixed a_{11} , it is easy to see that \tilde{h} attains its maximum for $a_{22} = a_{11}$. In this case, the optimal Q_d is given by

$$Q_d = \frac{1}{2} \left(a_{11} + \frac{a_{11}}{K}, Ka_{11} + a_{11} \right),$$

and

$$\max_{a_{22}} \tilde{h}(A_d) = -\frac{a_{11}^2}{4}(1 - 1/K)(K - 1).$$

Therefore (3.10) has a solution whenever

$$(3.11) \quad -\frac{a_{11}^2}{4}(1 - 1/K)(K - 1) < a_{21}a_{12}.$$

From (3.5) it follows that

$$|a_{21}| < \max\{S_1, S_2\}K\delta a_{22},$$

and hence

$$|a_{12}a_{21}| < \max\{S_1, S_2\}K\delta a_{22}|a_{12}| < \max\{S_1, S_2\}K\delta^2 a_{11}a_{22} < \max\{S_1, S_2\}K\delta^2 Ka_{11}^2.$$

By the very definition (3.4) of δ we deduce that

$$\frac{a_{11}^2}{4}(1 - 1/K)(K - 1) > \max\{S_1, S_2\}K^2\delta^2 a_{11}^2,$$

so that (3.11) holds, and the proof is completed. \square

4. TWO PHASE BELTRAMI COEFFICIENTS

In the present section we focus on two-phase Beltrami coefficients. In this class, we find the ellipticity K^{\min} defined in (2.25) and we characterize the Beltrami coefficients for which $K = K^{\min}$. From now on, to easy notation, we will omit the dependence on G and H in the ellipticity constants.

4.1. Two-phase Beltrami equation. Let E_1 be a measurable subset of Ω and let $E_2 := \Omega \setminus E_1$. Fix $\{G_1, G_2, H_1, H_2\} \subset SL_{sym}(2)$ positive definite (symmetric and with determinant one), and consider the functions

$$(4.1) \quad G := \chi_{E_1} G_1 + \chi_{E_2} G_2, \quad H := \chi_{E_1} H_1 + \chi_{E_2} H_2,$$

where χ_{E_1} and χ_{E_2} are the characteristic functions of E_1 and E_2 , respectively. From (2.11) it follows that for G and H of the form (4.1), one has

$$K = \max\{|gh|_{\mathbf{L} E_1}, |gh|_{\mathbf{L} E_2}\} = \max\{g_1 h_1, g_2 h_2\}.$$

where g_i and h_i denote the largest eigenvalue in E_i of G and H , respectively. Set

$$\hat{K} := \sqrt{g_1 h_1 g_2 h_2}.$$

Lemma 4.1. *The following inequality holds*

$$K^{\min} \leq \hat{K} \leq K.$$

Proof. The inequality $\hat{K} \leq K$ is trivial. Let us prove that $K^{\min} \leq \hat{K}$. Without loss of generality we may assume that $g_1 h_1 \geq g_2 h_2$. Set

$$\lambda := \sqrt{\frac{g_2 h_2}{g_1 h_1}} \leq 1.$$

We can have either of the following cases: $h_1 < \max\{g_1, g_2, h_1, h_2\}$ or $h_1 = \max\{g_1, g_2, h_1, h_2\}$. Suppose we are in the first case. Up to a diagonalization, G_1 is of the form

$$G_1 = \begin{pmatrix} g_1 & 0 \\ 0 & \frac{1}{g_1} \end{pmatrix}.$$

We want to use the change of variables (2.23), and we recall that $g(A, B)$ and $h(A, B)$ denote the maximum eigenvalue of \tilde{G} and \tilde{H} , respectively. We choose

$$B = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix},$$

and $A = I$. Then $g_1(A, B) = \lambda g_1$ and $g_2(A, B) \leq \frac{1}{\lambda} g_2$. Therefore

$$g_1(A, B)h_1(A, B) = \hat{K} \quad \text{and} \quad g_2(A, B)h_2(A, B) \leq \hat{K}.$$

We deduce

$$K^{\min} \leq g_1(A, B)h_1(A, B) = \hat{K}.$$

Suppose now that $h_1 = \max\{g_1, g_2, h_1, h_2\}$. Then, after diagonalization of H_1 , we choose $B = I$ and $A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$, and we proceed as before. \square

Remark 4.2. A direct consequence of Lemma 4.1 is that $K^{\min} < K$ whenever $g_1 h_1 < g_2 h_2$.

Proposition 4.3. *The following formula for K^{\min} holds:*

$$(4.2) \quad K^{\min}(G, H) = \sqrt{g_2(G_1^{-1/2}, H_1^{-1/2})h_2(G_1^{-1/2}, H_1^{-1/2})}.$$

Proof. In view of Lemma 4.1, it is enough to prove that for each $A, B \in SL(2)$ we have

$$(4.3) \quad g_2(G_1^{-1/2}, H_1^{-1/2})h_2(G_1^{-1/2}, H_1^{-1/2}) \leq g_1(A, B)h_1(A, B)g_2(A, B)h_2(A, B).$$

For this purpose, we show that if $G_1 = H_1 = Id$, then for each $A, B \in SL(2)$

$$(4.4) \quad g_2 \leq g_1(A, B)g_2(A, B),$$

$$(4.5) \quad h_2 \leq h_1(A, B)h_2(A, B).$$

Let $B \in SL(2)$ and set

$$\tilde{G}_1 := B^T B, \quad \tilde{G}_2 := B^T G_2 B.$$

For every $v \in \mathbb{R}^2$ we have

$$\frac{1}{g_1(A, B)} \|v\|^2 \leq \langle \tilde{G}_1 v, v \rangle = \|Bv\|^2,$$

and hence

$$g_2(A, B) = \sup_{\|v\| \leq 1} \langle \tilde{G}_2 v, v \rangle = \sup_{\|v\| \leq 1} \langle G_2 B v, B v \rangle \geq \frac{g_2}{g_1(A, B)},$$

which proves (4.4). The proof of (4.5) is fully analogous. \square

4.2. Gradient integrability and critical coefficients. In the next proposition we will show that if the bound $K^{\min} \leq K$ is achieved, then G_i and H_i can be simultaneously diagonalized.

Proposition 4.4. *Let G and H be as in (4.1) and assume that $K^{\min} = \hat{K}$. Then, there exist $A, B \in O(2)$ such that*

$$(4.6) \quad A^T G_1 A := \begin{pmatrix} g_1 & 0 \\ 0 & \frac{1}{g_1} \end{pmatrix}, \quad A^T G_2 A := \begin{pmatrix} \frac{1}{g_2} & 0 \\ 0 & g_2 \end{pmatrix},$$

$$(4.7) \quad B^T H_1 B := \begin{pmatrix} h_1 & 0 \\ 0 & \frac{1}{h_1} \end{pmatrix}, \quad B^T H_2 B := \begin{pmatrix} \frac{1}{h_2} & 0 \\ 0 & h_2 \end{pmatrix}.$$

Proof. We can always assume that G_1 and H_1 are as in (4.6)-(4.7). We prove that, in this case, also G_2 is diagonal (For H_1 and H_2 we argue exactly in the same way). Set

$$\begin{aligned} \hat{B} &:= G_1^{-\frac{1}{2}}, & \hat{G}_1 &:= \hat{B} G_1 \hat{B} = I, & \hat{G}_2 &:= \hat{B} G_2 \hat{B}, \\ \hat{A} &:= H_1^{-\frac{1}{2}}, & \hat{H}_1 &:= \hat{A} H_1 \hat{A} = I, & \hat{H}_2 &:= \hat{A} H_2 \hat{A}. \end{aligned}$$

Since $\hat{h}_2 \leq h_1 h_2$, $\hat{g}_2 \leq g_1 g_2$ and recalling Proposition 4.3 we have

$$(K^{\min})^2 = \hat{g}_1 \hat{h}_1 \hat{g}_2 \hat{h}_2 = \hat{g}_2 \hat{h}_2 \leq \hat{g}_2 h_1 h_2 \leq h_1 h_2 g_1 g_2 = \hat{K}^2,$$

where \hat{g}_i and \hat{h}_i are the largest eigenvalues of \hat{G}_i and \hat{H}_i . Since $K^{\min} = \hat{K}$, all the above inequalities are indeed equalities, and in particular $\hat{g}_2 = g_1 g_2$, that implies G_2 diagonal.

We are left to show that e_2 is the eigenvector associated with g_2 . Arguing by contradiction, we assume that

$$G_2 = \begin{pmatrix} g_2 & 0 \\ 0 & \frac{1}{g_2} \end{pmatrix}.$$

Without loss of generality we may suppose that $g_1 \leq g_2$ and we set

$$\hat{B} := G_1^{-\frac{1}{2}}, \quad \hat{G}_1 := \hat{B} G_1 \hat{B} = I, \quad \hat{G}_2 := \hat{B} G_2 \hat{B}.$$

It can be easily checked that $\hat{g}_i < g_i$, that (recall $K^{\min} = \hat{K}$) provides the following contradiction

$$(K^{\min})^2 \leq \hat{g}_1 h_1 \hat{g}_2 h_2 < g_1 h_1 g_2 h_2 = \hat{K}^2. \quad \square$$

We are in a position to show that, for two phase coefficients, Proposition 2.6 is sharp.

Theorem 4.5. *Let $G_1, G_2, H_1, H_2 \in SL_{\text{sym}}(2)$, and let K^{\min} be as defined in (4.2). Then we have*

- i) *Let G and H be as in (4.1). Every solution $f \in H_{\text{loc}}^1(\Omega; \mathbb{C})$ to (1.4) belongs to $L_{\text{loc}}^p(\Omega; \mathbb{C})$ for every $p \in [2, p_{K^{\min}})$;*
- ii) *There exist G and H as in (4.1), and a corresponding solution $f \in H_{\text{loc}}^1(\Omega; \mathbb{C})$ to (1.4) with $\nabla f \notin L^{p_{K^{\min}}}(B; \mathbb{M}^{2 \times 2})$ for every disk $B \subset \Omega$.*

Proof. The first part of the Theorem is a particular case of Proposition 2.6, so we pass to the proof of ii). By the definition of K^{min} and by Proposition 4.4, we can always assume that G and H are diagonal as in (4.6), (4.7), with $g_i h_i = K^{min}$. A straightforward computation shows that the corresponding σ , defined according to (2.4), takes the form

$$\sigma_1 := \text{diag}(S_1, K^{min}), \quad \sigma_2 := \text{diag}\left(S_2, \frac{1}{K^{min}}\right), \quad K^{-1} \leq S_i \leq K.$$

Therefore, ii) follows from Theorem 3.2. \square

Remark 4.6. In ii) of Theorem 4.5, we can also enforce that f_1 satisfies suitable affine boundary conditions.

5. TWO PHASE CONDUCTIVITIES

In this part we study the gradient summability of σ -harmonic functions corresponding to two phase conductivities. Let E_1 be a measurable subset of Ω and let $E_2 := \Omega \setminus E_1$. We assume that both E_1 and E_2 have positive measure. Given positive definite matrices $\sigma_1, \sigma_2 \in \mathbb{M}^{2 \times 2}$, define

$$(5.1) \quad \sigma := \chi_{E_1} \sigma_1 + \chi_{E_2} \sigma_2.$$

Set

$$K^{min} = K^{min}(\sigma) := K^{min}(G(\sigma), H(\sigma)).$$

5.1. Main results and optimality of the bound (1.8). We can now rephrase Theorem 4.5 (see also Remark 4.6) in terms of the coefficient σ .

Theorem 5.1. *Let $\sigma_1, \sigma_2 \in \mathbb{M}^{2 \times 2}$ be positive definite.*

- i) *Let σ be a two phase conductivity as in (5.1). Every solution $u \in H_{loc}^1(\Omega)$ to (1.1) satisfies $\nabla u \in L_{loc}^p(\Omega)$ for every $p \in [2, p_{K^{min}})$;*
- ii) *There exist σ as in (5.1) and $(v_1, v_2) \in \mathbb{R}^2$ such that the solution $u \in H_{loc}^1(\Omega)$ to*

$$\begin{cases} \text{div}(\sigma \nabla u) = 0 & \text{in } \Omega \\ u(x) = v_1 x_1 + v_2 x_2 & \text{on } \partial\Omega \end{cases}$$

satisfies for every ball $B \subset \Omega$

$$\int_B |\nabla u|^{p_{K^{min}}} dx = \infty.$$

We are in a position to prove that the bound in (1.8) is achieved by a suitable conductivity σ of the type

$$(5.2) \quad \sigma = \chi_{E_1} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \chi_{E_2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with } a = \lambda, b = \pm \sqrt{1 - \lambda^2}.$$

Theorem 5.2. *There exist σ as in (5.2), and a corresponding solution $u \in H_{loc}^1(\Omega)$ of (1.1) with affine boundary conditions such that $\nabla u \notin L^{p_{K_\lambda}}(B)$ for every disk $B \subset \Omega$, where p_{K_λ} is given by (2.22).*

Proof. By (2.3) we have $G_i(\sigma) = I$ for $i = 1, 2$, and

$$H_1 = \begin{pmatrix} \lambda^{-1} & \sqrt{1-\lambda^2} \\ \sqrt{1-\lambda^2} & \lambda^{-1} \end{pmatrix}, \quad H_2 = \begin{pmatrix} \lambda^{-1} & -\sqrt{1-\lambda^2} \\ -\sqrt{1-\lambda^2} & \lambda^{-1} \end{pmatrix}.$$

Therefore $K(\sigma) = K^{\min}(\sigma) = \frac{1+\sqrt{1-\lambda^2}}{\lambda} = K_\lambda$. We conclude in view of Theorem 5.1. \square

Finally, we fix the ellipticity λ and we characterize the pairs (σ_1, σ_2) corresponding to solutions with critical gradient integrability.

Theorem 5.3. *Let p_{K_λ} and $p_{K_\lambda^{\text{sym}}}$ be as in (2.22).*

i) *Let $\sigma \in \mathcal{M}(\lambda, \Omega)$ be a two phase conductivity as in (5.1) such that there exists a solution $u \in H^1(\Omega)$ of (1.1) with $\nabla u \notin L_{\text{loc}}^{p_{K_\lambda}}(\Omega; \mathbb{R}^2)$; then σ takes the following form*

$$\sigma = \chi_{E_1} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \chi_{E_2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \text{with } a = \lambda, b = \pm\sqrt{1-\lambda^2}.$$

ii) *Let $\sigma \in \mathcal{M}_{\text{sym}}(\lambda, \Omega)$ be a two phase conductivity as in (5.1), such that there exists a solution $u \in H^1(\Omega)$ of (1.1) with $\nabla u \notin L_{\text{loc}}^{p_{K_\lambda^{\text{sym}}}}(\Omega; \mathbb{R}^2)$; then, up to a rotation, σ takes the following form*

$$\sigma = \chi_{E_1} \text{diag}(S_1, \lambda^{-1}) + \chi_{E_2} \text{diag}(S_2, \lambda), \quad \text{with } \lambda \leq S_1, S_2 \leq \lambda^{-1}.$$

Proof. i) From Proposition 2.2 it follows that $K \leq \frac{1+\sqrt{1-\lambda^2}}{\lambda} = K_\lambda$. On the other hand, i) of Theorem 5.1 yields $K^{\min} \geq K_\lambda$. Lemma 4.1 implies $K^{\min} = \hat{K} = K_\lambda$, thus yielding $g_i h_i = K^{\min}$ in both phases. Now apply Proposition 2.3 to conclude that i) holds true. ii) Again from Proposition 2.2, Theorem 5.1 and Lemma 4.1 we deduce that $K^{\min} = \hat{K} = \frac{1}{\lambda}$. The thesis follows from Proposition 4.4. \square

5.2. The explicit formula for K^{\min} . Here we give a direct formula for K^{\min} depending on σ_1 and σ_2 .

Proposition 5.4. *Let $\sigma_1, \sigma_2 \in \mathbb{M}^{2 \times 2}$ be positive definite. Denote by Σ_1 and Σ_2 the symmetric part of σ_1 and σ_2 respectively, and by d_1 and d_2 their determinant,*

$$\Sigma_i := \sigma_i^S, \quad d_i := \det \Sigma_i, \quad i = 1, 2.$$

Then,

$$(5.3) \quad K^{\min} = \sqrt{\frac{m + \sqrt{m^2 - 4d_1 d_2}}{2\sqrt{d_1 d_2}} \frac{n + \sqrt{n^2 - 4}}{2}},$$

where

$$m := (\sigma_2)_{11}(\sigma_1)_{22} + (\sigma_1)_{11}(\sigma_2)_{22} - \frac{1}{2} \left((\sigma_2)_{12} + (\sigma_2)_{21} \right) \left((\sigma_1)_{12} + (\sigma_1)_{21} \right),$$

$$n := \frac{1}{\sqrt{d_1 d_2}} \left[\det \sigma_1 + \det \sigma_2 - \frac{1}{2} \left((\sigma_1)_{21} - (\sigma_1)_{12} \right) \left((\sigma_2)_{21} - (\sigma_2)_{12} \right) \right].$$

If in addition $\sigma_1, \sigma_2 \in \mathbb{M}_{\text{sym}}^{2 \times 2}$, then (5.3) reduces itself to

$$K^{\min} = \max \left\{ \sqrt{\frac{1}{\lambda_1}}, \sqrt{\lambda_2} \right\},$$

where $\lambda_1 \leq \lambda_2$ are the eigenvalues of $\sigma_1^{-1/2} \sigma_2 \sigma_1^{-1/2}$.

Proof. From Proposition 4.3 it follows that $K^{min} = \sqrt{g_2 h_2}$ where g_2 and h_2 are the maximum eigenvalues of $\tilde{G}_2 := G_1^{-1/2} G_2 G_1^{-1/2}$ and $\tilde{H}_2 := H_1^{-1/2} H_2 H_1^{-1/2}$ respectively. Since by (2.3), $G_i = \frac{1}{\sqrt{d_i}} \text{Adj} \Sigma_i$, one has

$$\tilde{G}_2 = \frac{\sqrt{d_1}}{\sqrt{d_2}} J \Sigma_1^{-1/2} \Sigma_2 \Sigma_1^{-1/2} J^T.$$

The eigenvalues of \tilde{G}_2 are solutions to the following equation in λ

$$\det \left(\sqrt{\frac{d_1}{d_2}} \Sigma_2 - \lambda \Sigma_1 \right) = 0.$$

Set $M := \Sigma_2 \text{Adj} \Sigma_1$. Since

$$\det \left(\sqrt{\frac{d_1}{d_2}} \Sigma_2 - \lambda \Sigma_1 \right) = 0 \iff \lambda^2 - \frac{\text{tr} M}{\sqrt{d_1 d_2}} \lambda + 1 = 0,$$

the maximum eigenvalue g_2 is defined by

$$g_2 = \frac{\frac{\text{tr} M}{\sqrt{d_1 d_2}} + \sqrt{\frac{(\text{tr} M)^2}{d_1 d_2} - 4}}{2}.$$

A straightforward computation shows that

$$\text{tr} M = (\sigma_2)_{11}(\sigma_1)_{22} + (\sigma_1)_{11}(\sigma_2)_{22} - \frac{1}{2} \left((\sigma_2)_{12} + (\sigma_2)_{21} \right) \left((\sigma_1)_{12} + (\sigma_1)_{21} \right) =: m.$$

Similarly, one finds that h_2 is the largest root of the equation

$$\det(H_2 - \lambda H_1) = 0.$$

Therefore

$$h_2 = \frac{\text{tr} N + \sqrt{(\text{tr} N)^2 - 4}}{2},$$

where $N := H_2 \text{Adj} H_1$. It is easily checked that

$$\text{tr} N = \frac{1}{\sqrt{d_1 d_2}} \left[\det \sigma_1 + \det \sigma_2 - \frac{1}{2} \left((\sigma_1)_{21} - (\sigma_1)_{12} \right) \left((\sigma_2)_{21} - (\sigma_2)_{12} \right) \right] =: n.$$

Now assume that σ_1, σ_2 are symmetric. By (2.20) we find $g_2 h_2 = \max \left\{ \frac{1}{\lambda_1}, \lambda_2 \right\}$, where $\lambda_1 \leq \lambda_2$ are the eigenvalues of

$$\tilde{\sigma}_2 := \frac{1}{(\tilde{H}_2)_{22}} \tilde{G}_2^{-1}.$$

Since by (2.3), $G_i = \frac{1}{\sqrt{\det \sigma_i}} \text{Adj} \sigma_i$, one has

$$\begin{aligned} \tilde{\sigma}_2 &= \frac{1}{(\tilde{H}_2)_{22}} \frac{\sqrt{\det \sigma_2}}{\sqrt{\det \sigma_1}} J \sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2} J^T \\ &= \frac{1}{(\tilde{H}_2)_{22}} \frac{1}{\sqrt{\det(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2})}} \text{Adj}(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2}) \\ &= \frac{1}{(\tilde{H}_2)_{22}} \sqrt{\det(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2})} (\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2})^{-1}. \end{aligned}$$

The eigenvalues of $\tilde{\sigma}_2$ are those of $\sigma_1^{-1/2} \sigma_2 \sigma_1^{-1/2}$ as soon as we prove that

$$(\tilde{H}_2)_{22} = \sqrt{\det(\sigma_1^{1/2} \sigma_2^{-1} \sigma_1^{1/2})} = \sqrt{\frac{\det \sigma_1}{\det \sigma_2}}.$$

This follows from the fact that H_1 and H_2 are diagonal and therefore

$$(\tilde{H}_2)_{22} = \frac{(H_2)_{22}}{(H_1)_{22}} = \frac{\sqrt{\det \sigma_1}}{\sqrt{\det \sigma_2}}.$$

□

Remark 5.5. Keeping the notation of Proposition 5.4, if $\sigma_1, \sigma_2 \in \mathbb{M}_{sym}^{2 \times 2}$ are positive definite, a straightforward computation shows that

$$p_{K^{min}} = \frac{2}{1 - \min \left\{ \sqrt{\frac{1}{\lambda_1}}, \sqrt{\lambda_2} \right\}}.$$

6. SOME G -CLOSURE RESULTS REVISITED

Quasiconformal mappings appear in many branches of mathematics. Only rather recently they have shown their power in the theory of composites. In the composite material literature one of the typical goals is to determine the so-called “ G -closure of a set of conductivities”. Roughly speaking this means the following. Assume that two matrices, called the conductivity of the “phases” and denoted by $\sigma_1, \sigma_2 \in \mathcal{M}(\lambda, \Omega)$ are given. Consider a two phase composites, i.e. a conductivity σ of the form $\sigma = \sigma_1 \chi_{E_1} + \chi_{E_2} \sigma_2$ where E_1 and E_2 are a pair of disjoint measurable sets with $E_1 \cup E_2 = \Omega$. The task is to find the set of all possible “effective” tensors σ^* that can be obtained by mixing these two phases while letting E_1 and E_2 vary in all the admissible ways. To make this concept precise, one needs to define an appropriate concept which is called H -convergence and was invented by Murat and Tartar. This notion was a general framework which was necessary to treat the case non-symmetric conductivity σ which could not be treated by the G -convergence previously introduced by Spagnolo. In both cases one can establish compactness results and a notion of closure. We will continue to call it G -closure according to tradition even if, in this particular case, one really needs to use the H -convergence because the tensor σ is not assumed to be symmetric a priori. We refer to the recent book of Tartar [18] and reference therein for an extensive treatment.

In this context, an extensive use of certain special properties of solutions to (1.1) and therefore to (2.7), has been made. For an accurate review, we refer to [16], see Chapter 4. As a particularly interesting case, we consider Milton’s work computing the so called G -closure of a mixture of two materials with arbitrary volume fractions [14]. In the symmetric case, i.e. when both phases have a symmetric conductivity, the G -closure was found in the eighties. The result has a long history which is reviewed in a very recent work by Francfort and Murat [9]. We refer the reader to the reference therein for more details about the original work.

Milton studied the general case without assuming symmetry. He proved that one can recover the G -closure for this case by first reducing the problem to the study of a two-phase composite in which, in addition, each phase is symmetric, [14] and Chapter 4.3 in [16], and then applying the results for the symmetric case. Milton explained how his work was generalizing previous work by many authors including Keller, Dykhne, Mendelsohn and that, in turn, he was inspired by some work of Francfort and Murat and some unpublished work by Tartar now available in [18], Lemma 20.3: in two dimensions “homogenization commutes with certain Moebius transformations”. Without entering into too many details, we want

to emphasize here that the basic ingredients behind these transformations have an elegant geometrical counterpart when expressed in terms of the Beltrami equation.

When σ is two-phase, by (2.3), so are the matrices H and G . In particular $H = H_1\chi_1 + H_2\chi_2$. Consider now the equation (2.1) and make the affine change of variable $f \rightarrow F = Af$, then F satisfies a new equation in which the matrix H is replaced by $H_A := A^T H A / (\det A)$. Therefore choosing $A = H^{-\frac{1}{2}} R_2^T$ with $R_2 \in SO(2)$ and such that $R_2^T H_2 R =: D_2$ is diagonal, one has

$$(6.1) \quad H_A = I\chi_1 + D_2\chi_2$$

so that H_A is diagonal and thus $(H_A)_{12}$ is identically zero. This in turn implies, by (2.4) that the corresponding conductivity

$$\sigma_A := \frac{G^{-1} + (H_A)_{12}J}{(H_A)_{22}}$$

is symmetric.

We observe, in passing, that applying the same strategy to the domain of f one can independently reduce a two-phase G to the form

$$(6.2) \quad G_B = I\chi_1 + G_2\chi_2$$

with G_2 a diagonal matrix by a linear transformation $x \rightarrow Bx$.

In the work of Milton, the “symmetrization” property for a two-phase composites is obtained as follows. Let $\lambda \in [0, 1]$ and let $\sigma \in \mathcal{M}(\lambda, \Omega)$. Set

$$(6.3) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \Sigma_A = (a\sigma + bJ)(cI + dJ\sigma)^{-1}$$

and let $U_\sigma = (u_\sigma^1, u_\sigma^2)$ be any solution to the equation (2.7) i.e. $\sigma \nabla u_\sigma^1 = J^T \nabla u_\sigma^2$.

Proposition 6.1. *For any two-phase composites, there exists A as in (6.3) such that the corresponding Σ_A is symmetric and moreover for some $\lambda' \in [0, 1]$ one has $\Sigma_A \in \mathcal{M}(\lambda', \Omega)$.*

To continue the argument Milton needs to prove that the G -closure problem relative to Σ_A is mapped one to one into that relative to σ . He uses [14] the commutation of the linear fractional transformation $\sigma \rightarrow \Sigma_A$ with homogenization, see also [18], Lemma 20.3.

Our perspective is to use the following property.

Proposition 6.2. *For any given A as in (6.3) for which $\Sigma_A \in \mathcal{M}(\lambda', \Omega)$ for some $\lambda' \in [0, 1]$, there exists*

$$(6.4) \quad A' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

such that any solution $U_{\Sigma_A} = (u_{\Sigma_A}^1, u_{\Sigma_A}^2)$ to $\Sigma_A \nabla u_{\Sigma_A}^1 = J^T \nabla u_{\Sigma_A}^2$ takes the form

$$(6.5) \quad U_{\Sigma_A} = A' U_\sigma.$$

Proof. We need to prove that there exist $\{a', b', c', d'\}$ such that

$$\Sigma_A(a' \nabla u_\sigma^1 + b' \nabla u_\sigma^2) = J^T(c' \nabla u_\sigma^1 + d' \nabla u_\sigma^2),$$

which is equivalent to show that

$$(a' \Sigma_A - c' J^T) \nabla u_\sigma^1 + (b' \Sigma_A - d' J^T) \nabla u_\sigma^2 = 0.$$

We now use the equation $\sigma \nabla u_\sigma^1 = J^T \nabla u_\sigma^2$ and write the previous equation as

$$[a' \Sigma_A - c' J^T + (b' \Sigma_A - d' J^T) J \sigma] \nabla u_\sigma^1 = 0.$$

One possible solution (actually the only one) is found if the matrix in square brackets is zero i.e. if and only if

$$\begin{aligned} a' \Sigma_A - c' J^T + (b' \Sigma_A - d' J^T) J \sigma &= 0 \Leftrightarrow \Sigma_A (a' I + b' J \sigma) = c' J^T + d' \sigma \Leftrightarrow \\ \Sigma_A &= (c' J^T + d' \sigma) (a' I + b' J \sigma)^{-1} \end{aligned}$$

and the latter is equivalent to make the following choice:

$$(6.6) \quad A' = \begin{pmatrix} c & d \\ -b & a \end{pmatrix}.$$

□

Proposition 6.2 is the key property to the commuting rule and it is, indeed, a linear change of variables in the target space of the underlying quasiregular mapping $U = (u, v)$, solution to (2.7).

Finally one may wonder whether (6.6) can be chosen in such a way to have $\Sigma_A \in \mathcal{M}(\lambda', \Omega)$ for some $\lambda' > 0$. To check this we first note that

$$\begin{aligned} \Sigma_A &= (a\sigma + bJ)(cI + dJ\sigma)^{-1} = \frac{a\sigma + bJ}{c^2 \det \sigma + d^2} \text{Adj}(cI + dJ\sigma) = \\ &= \frac{a\sigma + bJ}{c^2 \det \sigma + d^2} (cI + dJ^T (\sigma^T J^T) J) = \frac{a\sigma + bJ}{c^2 \det \sigma + d^2} (cI + dJ^T \sigma^T) = \\ &= \frac{ac\sigma + bcJ + ad\sigma J^T \sigma^T + bd\sigma^T}{c^2 \det \sigma + d^2} = \frac{ac\sigma + bd\sigma^T + bcJ + ad \det \sigma J^T}{c^2 \det \sigma + d^2}. \end{aligned}$$

It follows that

$$(6.7) \quad (\Sigma_A)^S = \frac{\Sigma_A + \Sigma_A^T}{2} = \frac{ac + bd}{c^2 \det \sigma + d^2} \sigma^S.$$

Therefore, recalling (6.6), the first necessary condition to (1.2) can be expressed as follows

$$(6.8) \quad c^2 + d^2 > 0 \quad , \quad ac + bd > 0 \Leftrightarrow ac + bd > 0 \Leftrightarrow \det A' > 0.$$

Now we need to consider Σ_A^{-1} .

$$\begin{aligned} \Sigma_A^{-1} &= (cI + dJ\sigma)(a\sigma + bJ)^{-1} = \frac{cI + dJ\sigma}{a^2 \det \sigma + b^2} \text{Adj}(a\sigma + bJ) = \\ &= \frac{cI + dJ\sigma}{a^2 \det \sigma + b^2} (aJ\sigma^T J^T + bJ^T) = \frac{(cI + dJ\sigma)(aJ\sigma^T J^T + bJ^T)}{a^2 \det \sigma + b^2} = \\ &= \frac{acJ\sigma^T J^T + adJ\sigma J\sigma^T J^T + bcJ^T + bdJ\sigma J^T}{a^2 \det \sigma + b^2} = \\ &= \frac{acJ\sigma^T J^T + bdJ\sigma J^T + adJ^T \det \sigma + bcJ^T}{a^2 \det \sigma + b^2}. \end{aligned}$$

It follows that

$$(6.9) \quad (\Sigma_A^{-1})^S = \frac{ac + bd}{a^2 \det \sigma + b^2} J \sigma^S J^T.$$

Therefore the second necessary condition to (1.3) is expressed as follows

$$(6.10) \quad a^2 + b^2 > 0, \quad ac + bd > 0 \iff ac + bd > 0 \iff \det A' > 0.$$

Putting (6.8) and (6.10) together we obtain

$$(6.11) \quad \Sigma_A \in \mathcal{M}(\lambda', \Omega) \quad \text{for some} \quad \lambda' > 0 \iff \det A' > 0.$$

Again, this fact has a clear interpretation in the language of the Beltrami system, recalling that A' represents a linear change of variables in the target space and that ellipticity in this context is measured according to Proposition 2.1.

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(V. Nesi) DIPARTIMENTO DI MATEMATICA “G. CASTELNUOVO”, SAPIENZA, UNIVERSITÀ DI ROMA, PIAZZALE A. MORO 2, 00185 ROMA, ITALY

E-mail address, V. Nesi: `nesi@mat.uniroma1.it`

(M. Palombaro) CENTRO DE GIORGI, SCUOLA NORMALE SUPERIORE, PIAZZA DEI CAVALIERI 3, 56126 PISA, ITALY

E-mail address, M. Palombaro: `mariapia.palombaro@sns.it`

(M. Ponsiglione) DIPARTIMENTO DI MATEMATICA “G. CASTELNUOVO”, SAPIENZA, UNIVERSITÀ DI ROMA, PIAZZALE A. MORO 2, 00185 ROMA, ITALY

E-mail address, M. Ponsiglione: `ponsigli@mat.uniroma1.it`